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A Scheme for the Numerical Solution of Hyperbolic Systems of Conservation Laws

W. D. HENSHAW[1]

Department of Applied Mathematics 217-50, California Institute of Technology, Pasadena, California 91125

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A method is described for the numerical solution of hyperbolic systems of conservation laws in one space dimension. The basis of the scheme is to use finite differences where the solution is smooth and the method of characteristics where the solution is not smooth. The method can accurately represent shocks. Results are presented for the solution of the equations of gas dynamics. The examples illustrate the accuracy of the method when discontinuities are present and the code's performance on difficult problems of interacting shocks and shock formation.

1. INTRODUCTION

We consider the numerical solution of hyperbolic systems of conservation laws. Such problems can be difficult to solve numerically since the solutions can exhibit discontinuities. However, there are many problems which can be cast into this form and thus it is of some importance to develop good numerical schemes. The approach taken here is to use a hybrid method which combines finite difference methods with the method of characteristics. Finite differences are easy to implement and accurate when the numerical solution is smooth. The method of characteristics is more difficult to implement but is accurate when there are discontinuities present. The idea is to combine the methods, using finite differences where the solution is smooth and using the method of characteristics otherwise. The finite difference method is applied on a fixed grid. The method of characteristics is used on points which move through the fixed grid. The position and number of these *characteristic* points may vary with time. Shocks appear as perfect discontinuities. They are recognized by the crossing of characteristics and are fitted using the shock relations. Interactions between different shocks are handled in a uniform manner by the use of a Riemann solver.

The method described here is a generalized shock tracking approach. The basics of shock tracking is described in Richtmyer and Morton [10]. Applications of shock tracking in one and two space dimensions are described in Glimm *et al.* [1],

[1] Present address: IBM, Thomas J. Watson Research Center, P.O. Box 218, Yorktown Heights, NY 10598.

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DeNeef and Hechtman [3], Lötstedt [9], Salas [11], and Zhu *et al.* [15], among others.

1.1. *Background*

A system of hyperbolic conservation laws in one space dimension can be written in the form

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = 0. \tag{1.1}$$

Here $\mathbf{u}: \mathbb{R} \times [0, \infty) \to \mathbb{R}^m$ is a vector with m components, each component being a real valued function of x and t. $\mathbf{f}$ is called the flux function, $\mathbf{f}: \mathbb{R}^m \to \mathbb{R}^m$. The transpose of a vector $\mathbf{u}$ will be denoted as $\mathbf{u}^{\mathrm{T}}$. The system (1.1) is said to be hyperbolic if the eigenvalues $\{c_i(\mathbf{u})\}_{i=1}^m$ of the Jacobian matrix

$$J(\mathbf{u}) = \mathbf{f}_{\mathbf{u}} = \left[\frac{\partial f_i}{\partial u_j}\right]$$

are real and there is a complete set of eigenvectors. It will be assumed here that the eigenvalues are distinct and can be ordered

$$c_1 < c_2 < c_3 < \cdots < c_m.$$

Let $\mathbf{a}_i(\mathbf{u})$ denote the left eigenvector of J corresponding to the eigenvalue c_i, $\mathbf{a}_i^{\mathrm{T}} J = c_i \mathbf{a}_i^{\mathrm{T}}$. Multiplying the conservation equation (1.1) by $\mathbf{a}_i(\mathbf{u})^{\mathrm{T}}$ and using the eigenvalue equation gives

$$\mathbf{a}_i^{\mathrm{T}} \left[\frac{\partial \mathbf{u}}{\partial t} - c_i(\mathbf{u}) \frac{\partial \mathbf{u}}{\partial x}\right] = 0.$$

Each of these equations reduces to an ordinary differential equation along the characteristic curve C_i whose slope in $x-t$ space is $c_i(\mathbf{u})$,

$$\mathbf{a}_i^{\mathrm{T}} \frac{d\mathbf{u}}{dt} = 0 \qquad \text{along } C_i: \frac{dx}{dt} = c_i(\mathbf{u}), \qquad i = 1, 2, \ldots, m. \tag{1.2}$$

These are known as the characteristic equations.

If $\mathbf{f}$ is a nonlinear function of $\mathbf{u}$ then in general classical solutions to the initial value problem do not exist for all time; see for example Whitham [14]. Derivatives of $\mathbf{u}$ can become infinite in a finite time even for smooth initial data. Often systems such as (1.1) describe the limiting behaviour of a physical process as some parameter goes to zero. For example, the equations of gas dynamics to be discussed later are the limiting equations as the effects of viscosity and heat conduction go to zero. The breakdown of the solution may then be related to the breakdown of some of the assumptions under which the equations were derived. To obtain the physically meaningful solution one could solve a new set of equations which includes those effects that are now important. For example, one often really wants the solution to a related viscous problem

$$\mathbf{u}_t + \mathbf{f}(\mathbf{u})_x = \varepsilon(\beta(\mathbf{u})\, \mathbf{u}_x)_x, \qquad \beta(\mathbf{u}) \geqslant 0$$

as the viscosity ε tends to zero. Solving these equations accurately can be much more work since one must resolve the shocks. In many cases the structure through the shock is not required. As an alternative it is possible to patch up the current set of equations by extending the notion of what is meant by a solution. This is done by allowing the solution to have discontinuities. At a propagating discontinuity, on either side of which the solution is continuously differentiable, one can appeal to the integral form of the conservation laws to obtain the equations which describe how the discontinuity or *shock* is to be propagated. These are the Rankine–Hugoniot shock relations

$$[\mathbf{f}(\mathbf{u}_R) - \mathbf{f}(\mathbf{u}_L)] = U[\mathbf{u}_R - \mathbf{u}_L]. \tag{1.3}$$

U is the speed of propagation of the discontinuity. $\mathbf{u}_R$ and $\mathbf{u}_L$ are the states to the right and left of the shock. One way to mathematically define a solution to (1.1) which allows for discontinuities is to introduce the concept of a generalized solution. We call $\mathbf{u}$ a generalized or weak solution of the system (1.1) with initial conditions $\mathbf{u}(x, 0) = \mathbf{u}_0(x)$ if for all smooth test functions $\phi(x, t)$ of compact support

$$\int_{t=0}^{\infty} \int_{x=-\infty}^{\infty} [\mathbf{u}\phi_t + \mathbf{f}(\mathbf{u})\,\phi_x]\,dx\,dt - \int_{x=-\infty}^{\infty} \mathbf{u}_0(x)\,\phi(x, 0)\,dx = 0. \tag{1.4}$$

This expression can be formally obtained in the following manner. Multiply the conservation equation (1.1) by ϕ and integrate over time and space. Integrate by parts to remove the derivatives from $\mathbf{u}$ and $\mathbf{f}$ and place them onto ϕ. This gives Eq. (1.4). Any classical solution of the conservation equation will thus be a generalized solution. The converse of this statement is not true. Having extended the solution space in this manner we run into the trouble that too many solutions are now allowed. We must use other criteria to determine which weak solution is the physically relevant one. This extra condition is called the *entropy condition*. For our purposes the entropy condition is simply the geometrical statement that the characteristics on either side of the discontinuity must run into (and not out of) the discontinuity. This means that for some index j

$$c_j(\mathbf{u}_L) > U > c_j(\mathbf{u}_R). \tag{1.5}$$

We further require that not too many characteristics run into the discontinuity so that

$$c_j(\mathbf{u}_L) > U > c_{j-1}(\mathbf{u}_L) \tag{1.6}$$

and

$$c_{j+1}(\mathbf{u}_R) > U > c_j(\mathbf{u}_R). \tag{1.7}$$

These conditions ensure that there are the correct number of equations to determine the evolution of the discontinuity. A propagating discontinuity satisfying the

entropy conditions (1.5)–(1.7) will be called a shock. If a discontinuity satisfies (1.5)–(1.7) with the inequalities in (1.5) replaced by equalities then it is called a *contact discontinuity*. There are alternative ways to define an entropy condition in terms of an entropy function (Lax [8]).

An important class of problems are those where the initial conditions consist of two constant states separated by a jump

$$\mathbf{u}(x,0)=\begin{cases}\mathbf{u}_L & (\text{constant})\ x<0\\ \mathbf{u}_R & (\text{constant})\ x>0.\end{cases}$$

These are known as Riemann problems. For many hyperbolic systems of interest the solution to the Riemann problem can be found. There are methods which are based upon the solution of the Riemann problem. These include Godunov's method [5] and Glimm's random choice method [4], among others. The method described in this paper also assumes that one can solve the Riemann problem. This knowledge is needed to handle the interaction of discontinuities.

These are a number of good references for further details of the material presented here, for example, Lax [8] and Whitham [14].

2. Description of the Scheme

The basic idea of the scheme has already been outlined in the Introduction. In this section we proceed to give a more complete description. In Section 2.1 the structure of the computational grid is discussed. By computational grid we refer to an underlying fixed grid together with extra grid points, called *characteristic points*. The number and positions of these extra points varies with time. The concept of a *group* of characteristic points is introduced and explained. Later sections describe how these groups of points are advanced in time. This involves a discussion of the method of characteristics, shock fitting, shock interactions and the Riemann problem.

2.1. *Grid Structure*

The computational grid is comprised of two types of grid points. First there are points which lie on a fixed grid. These points will be denoted by x_i and the corresponding solution values by $\mathbf{u}(i)$. For simplicity this grid is taken to have a constant mesh spacing h so that $x_{i+1}=x_i+h$. In addition to this uniform mesh there are also some extra points which move through the fixed grid as the solution develops. These extra points will be called *characteristic points* since they will be the points where the method of characteristics is applied. They will be located in regions where the solution is not smooth such as around discontinuities. Denote their positions by $x_c(i)$. The solution value a $x_x(i)$ will be called $\mathbf{u}_c(i)$. The characteristic points are not restricted to lie on any grid. Their positions are variable. In fact, when a shock develops two characteristic points will occupy the same location.

One of these points will carry the information for the state to the left of the shock and the other holds the state to the right of the shock. This allows shocks to be represented as true discontinuities. Figure 1 shows what the computational grid and a component of the solution might look like. The solution is the *shock tube* example of Section 4.5 and consists of two discontinuities, a shock to the right and a contact discontinuity in the center. There is an expansion fan to the left. The characteristic points and their solution values are marked with circles. Notice that there are two characteristic points located at the position of the shock and at the position the contact discontinuity.

Much of the complexity of the method is associated with the problem of keeping track of the characteristic points as they move through the fixed grid. Here is an outline of how this problem was resolved. Characteristic points will tend to cluster in certain locations, such as around shocks or around discontinuities in the first derivative. (In Fig. 1 there is a discontinuity in the first derivative at the edge of the expansion fan.) Such a cluster of characteristic points will be identified as a logical entity and will be called a *group*. Each group is separated from other groups by a smooth portion of the solution. Let $\mathbf{g}_i$ denote the ith group. It consists of a number of characteristic points

$$\mathbf{g}_i = \{x_c^i(j)\, j = 1,..., n_i\}.$$

A superscript i has been added to the characteristic point to denote that it belongs to group i. The smallest and largest values of $x_c^i(j)$ in any group define the extent of the group. Fixed grid points which lie *underneath* a group, that is within the extent of the group, will not be used. Such points will be called *inactive*, as opposed to

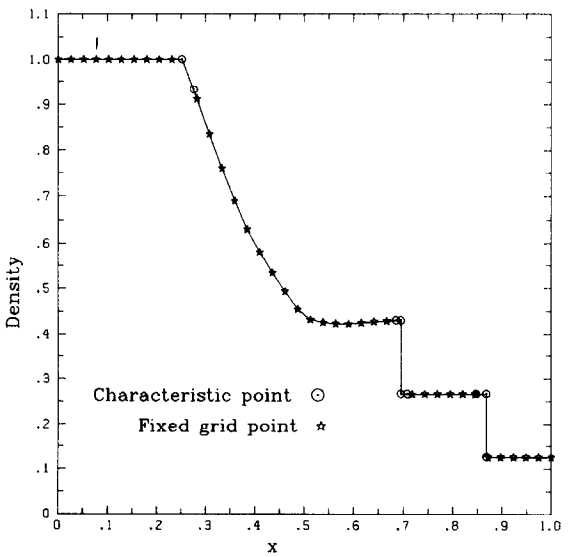

FIG. 1. Grid structure.

active fixed grid points which lie under no group. In Fig. 1 there are three groups, one around the shock, a second at the contact and the third around the corner of the expansion fan.

To advance the solution from one time step to the next the method of characteristics is applied to each group separately. Solution values outside the extent of the group may be needed. These can be obtained from neighbouring points on the fixed grid. The numerical implementation of the method of characteristics is described in Section 2.3. The fixed grid points are advanced using a finite difference method which is described in Section 2.2. The computational grid is monitored to make sure that all groups and all active fixed grid points satisfy certain conditions. The groups can be constantly changing in size and position. The following lists some of the operations that may be applied to a group or pair of groups.

(i) Merging—if two groups are too close together they are joined to form one new and larger group.

(ii) Splitting—if there is a smooth region in the interior of a group, the group is separated into two groups. Points on the fixed grid which lie in this smooth region become active points.

(iii) Liquidation—groups may disappear if the solution becomes smooth at all the points which make up the group.

(iv) Trimming—if there is a smooth region on the end of a group characteristic points will be taken away.

(v) Addition—Two adjacent characteristic points in a group are not allowed to get too far apart. Extra points may be added to prevent this from happening.

(vi) Creation—when the solution on the fixed grid becomes *rough* a new group may form. (Described in more detail below.)

New groups may appear spontaneously when the solution becomes *rough.* The smoothness of the solution on the fixed grid is measured by a normalized second undivided difference quotient. A point i on the fixed grid will become the location of a new characteristic point if

$$\max_{1 \leq j \leq m} \frac{|u_j(x_{i+1}) - 2u_j(x_i) + u_j(x_{i-1})|}{\|u_j\|} > \delta. \tag{2.1}$$

Here δ is a predetermined constant which will depend on h. $u_j(x_i)$ is the jth component of the solution at position x_i of the fixed grid and $\|u_j\|$ is a global measure of the size of the jth component of the solution. The left hand side of (2.1) will be $O(h^2)$ where the numerical solution is smooth with respect to the grid. If this quantity becomes large compared to h^2 then the finite difference method is likely losing accuracy and it is time to switch to the method of characteristics. The measure that one uses should be related to the accuracy of the finite difference method which is being used. A similar measure to (2.1) is used to determine when the numerical solution within a group is becoming smooth.

Programming is considerably simplified when the correct data structures are used. The data structure for holding the groups is straightforward in nature, consisting of pointers and lists. It is convenient to keep the characteristic points ordered by their position so that neighbours are easily found. A useful array to have, which simplifies many group operations, is one which indicates the *status* of each point on the fixed grid. The status will indicate whether the point is active or not and for inactive points will indicate which group it lies underneath of. Denoting this array by $istatus(i)$, say, then

$$istatus(i) = \begin{cases} 0 & \text{if fixed grid point } i \text{ is an active point} \\ k & \text{if point } i \text{ is inactive and sits below group } k. \end{cases}$$

Using this array it is easy to check whether two groups are getting close together. Groups are merged when they are less than a few mesh widths apart. In addition the array acts a system of pointers from the fixed grid to the data structure containing the groups.

2.2. *The Finite Difference Equations*

The numerical solution on the fixed grid is advanced using a finite difference method. The method that is used is the second order Lax–Wendroff scheme. Other difference schemes could be used. Since the scheme is only applied where the solution is smooth, higher order methods might prove to be useful.

2.3. *Solving the Characteristic Equations*

In this section we discuss the numerical solution of the characteristic equations which were derived in Section 1.1,

$$\mathbf{a}_i^{\mathrm{T}} \frac{d\mathbf{u}}{dt} = 0 \qquad \text{along} \quad C_i: \frac{dx}{dt} = c_i(\mathbf{u}), \qquad i = 1, 2, \ldots, m.$$

These equations are a coupled system of nonlinear ordinary differential equations. The system is not, however, in the standard form $d\mathbf{y}/dx = \mathbf{f}(x, \mathbf{y})$. Each characteristic equation only holds upon a curve whose position depends upon the solution. Given the solution $\mathbf{u}$ everywhere at time t the objective is to calculate the solution $\mathbf{u}$ for a particular point $(x, t + \Delta t)$. In the simplest case, when there are no shocks, there will be precisely m characteristic curves which intersect the point $(x, t + \Delta t)$. These characeristics emanate from some (unknown) points (x_i, t). The m characteristics carry enough information to determine the m unknown components of $\mathbf{u}(x, t + \Delta t)$.

To solve the equations numerically we proceed as follows. Suppose we know an approximation to the solution at all grid points at time t. Let $\mathbf{v}(x, t)$ denote the function which equals this solution at each grid point and varies linearly in between. Consider the task of determining the solution at some point z at time

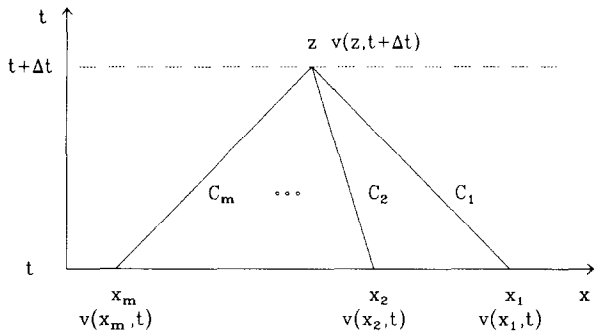

FIG. 2. Solving the characteristic equations.

$t + \Delta t$ by the method of characteristics (see Fig. 2). To do this the following second order implicit approximation to the characteristic equations is solved,

$$\begin{aligned} &\mathbf{a}_i(\tfrac{1}{2}(\mathbf{v}(z, t+\Delta t) + \mathbf{v}(x_i, t)))^{\mathrm{T}}(\mathbf{v}(z, t+\Delta t) - \mathbf{v}(x_i, t)) = 0, \\ &z - x_i = c_i(\tfrac{1}{2}(\mathbf{v}(z, t+\Delta t) + \mathbf{v}(x_i, t)))\,\Delta t, \end{aligned} \qquad i = 1, 2, \ldots, m. \quad (2.2)$$

There are $2m$ equations for the $2m$ unknowns

$$\begin{aligned} &\mathbf{v}(z, t+\Delta t) && m \text{ unknowns} \\ &x_i, && i = 1, 2, \ldots, m. \end{aligned}$$

These nonlinear equations are solved by a quasi-Newton iteration. The convergence of this iteration was considered in Henshaw [6].

In practice we may also want to solve the same equations when the initial position x_i of one of the characteristics is known and z is unknown.

As mentioned previously, each group of characteristic points is advanced as a unit. The steps to advance a group to the next time level (once shocks have been fitted) are as follows:

(i) Calculate $\mathbf{v}(z, t+\Delta t)$ as outlined above where z lies at the end of the jth characteristic curve which begins from a given characteristic point. Do this for each characteristic ($j = 1, 2, \ldots, m$) and for each point in the group.

(ii) Each characteristic point at time t has spawned m new points at time $t + \Delta t$. These points are not all kept; points are removed where the solution is smoothest.

3. DISCONTINUITIES

The solutions to hyperbolic systems of conservation laws can possess discontinuities. These may be present from time zero if the initial conditions contain

jumps. The discontinuities may also develop in time if the system is nonlinear. Discontinuities are treated in a special manner by the program.

Consider an isolated discontinuity which is propagating through the flow. The states on either side of the discontinuity are assumed to be smooth. The speed of the discontinuity, U, is then given by the jump conditions

$$[\mathbf{f}(\mathbf{u}_R) - \mathbf{f}(\mathbf{u}_L)] = U[\mathbf{u}_R - \mathbf{u}_L].$$

The discontinuity is assumed to satisfy the entropy condition. This condition was given in Section 1.1. In words the entropy condition states that there is one and only one extra characteristic running into the discontinuity. We will usually call a discontinuity which satisfies the jump conditions and the entropy condition a *shock*. The numerical procedure for advancing a shock (*shock fitting*) is described in Section 3.1. We will see that the jump conditions and the one extra characteristic equation provide enough equations to determine the shock speed and the states to the left and right of the shock.

Another type of discontinuity can exist at a given point in time. This discontinuity does not satisfy the jump conditions. It may arise in initial conditions or when two shocks collide. In order to determine the solution at the next time level a more general Riemann problem is solved. The discontinuity will then be resolved into shocks and expansion fans.

Before the characteristic points are advanced to the next time level, the group is first scanned for the existence of shocks. Shocks are indicated by the crossing of two characteristics of the same *family*, that is lying on the same numbered characteristic. The program considers two cases when characteristics are found to cross.

(i) Most often the crossing has occurred where a shock has been previously fitted. In this case the states on either side wil not be arbitrary but will satisfy the shock relations

$$[\mathbf{f}(\mathbf{u}_R) - \mathbf{f}(\mathbf{u}_L)] = U[\mathbf{u}_R - \mathbf{u}_L].$$

Actually since the shock velocity U is not known, the ratios of the jump in f_i to the jump in u_i are checked to see if they are the same. This also gives a good initial guess for U. The procedure for determining the position of the shock at the next time level is described in the next section.

(ii) If the conditions of (i) are not satisfied the discontinuity is not a shock. A Riemann problem can be solved to determine the solution at the next time step. A numerical technique for the solution of a general class of Riemann problems is discussed in Section 3.3.

3.1. *Fitting a Single Shock*

The shock fitting problem requires the determination of the states to the left and right of the shock at the next time level as well as the shock speed. The appropriate characteristic equations to use in determining these values are those which corres-

pond to the characteristic curves which do not cross the shock. Hence for a shock occurring on the characteristic family k (called a *k-shock*) there will be equations for characteristics 1 to k coming from the right of the shock and equations for characteristics k to m coming from the left of the shock (see Fig. 3).

There are $3m+2$ unknowns

$\mathbf{u}_L, \mathbf{u}_R$	states to the left and right of the shock
U	shock velocity
$x_1,..., x_k, y_k,..., y_m$	initial positions of the characteristics.

The $3m+2$ equations which must be solved are

$$\mathbf{a}_i(\tfrac{1}{2}(\mathbf{u}_R + \mathbf{v}(x_i, t)))^T(\mathbf{u}_R - \mathbf{v}(x_i, t)) = 0, \qquad i = 1, 2,..., k \tag{3.1}$$

$$\mathbf{a}_i(\tfrac{1}{2}(\mathbf{u}_L + \mathbf{v}(y_i, t)))^T(\mathbf{u}_L - \mathbf{v}(y_i, t)) = 0, \qquad i = k, k+1,..., m \tag{3.2}$$

$$[\mathbf{f}(\mathbf{u}_R) - \mathbf{f}(\mathbf{u}_L)] - U[\mathbf{u}_R - \mathbf{u}_L] = 0 \qquad (m \text{ equations}) \tag{3.3}$$

$$x_s - x_i = c_i(\tfrac{1}{2}(\mathbf{u}_R + \mathbf{v}(x_i, t)))\,\Delta t, \qquad i = 1, 2,..., k \tag{3.4}$$

$$x_s - y_i = c_i(\tfrac{1}{2}(\mathbf{u}_L + \mathbf{v}(y_i, t)))\,\Delta t, \qquad i = k, k+1,..., m. \tag{3.5}$$

The position of the shock at $t + \Delta t$ is x_s,

$$x_s \equiv x_s^0 + \tfrac{1}{2}[U + U^{(0)}]\,\Delta t.$$

In this last equation x_s^0 is the shock position at time t and $U^{(0)}$ is the shock velocity at time t. $U^{(0)}$ can be determined from the jump conditions. These equations are solved by a quasi-Newton method; by which to say that only the first $2m+1$ equations and variables are used in the Newton step. The variables x_i and y_i are updated after each Newton step from the equations which describe the position of the characteristic. This was done for simplicity.

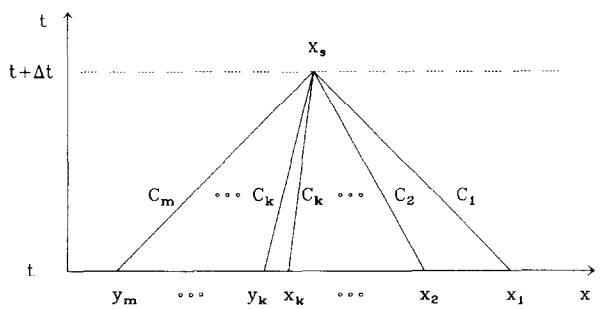

FIG. 3. Shock fitting.

3.2. *Shock Interactions*

The shock relations describe how isolated discontinuities propagate. However, this information does not yet completely specify the solution. For what happens when two shocks merge or collide? Further details must be given to indicate how the interactions between discontinuities are to be handled. Below we describe the manner in which shock interactions proceed.

Consider the situation when two shocks collide (see Fig. 4). The speeds of the shocks are simply determined by the states immediately in front of and in back of each shock. The shock speeds are given by the Rankine–Hugoniot relations. The shocks will continue to move together until the section between them vanishes. Now the states on either side of this discontinuity will not in general satisfy the jump conditions for a shock. This problem must then be considered as a general Riemann problem to be solved. The solution to this Riemann problem should generate the appropriate shocks, contacts and fans that result from the collision. It is not hard to show that the scenario given above for the collision of discontinuities describes a weak solution to the conservation laws.

3.3. *The General Riemann Solver*

In this section a numerical procedure for solving the Riemann problem is discussed. Efficient algorithms have been devised for specialized systems. An iteration procedure for the equations of gas dynamics with certain types of gas laws was given by Godunov [5]. Improvements and extensions of this scheme were made by, for example, van Leer [13] and Colella and Glaz [2]. The algorithm presented here applies to more general systems. It is assumed, however, that the solution is self-similar in the variable x/t and consists of constant states separated by shocks or fans (see Fig. 5). There is a good discussion of the Riemann problem in Lax [8].

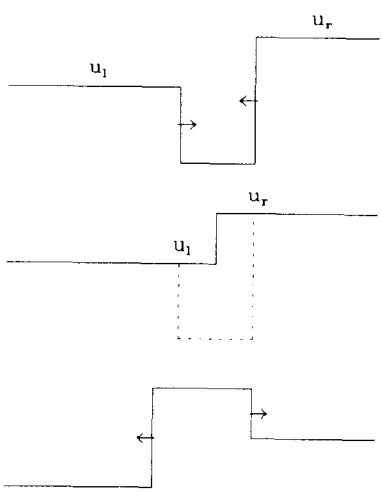

FIG. 4. Shock interactions.

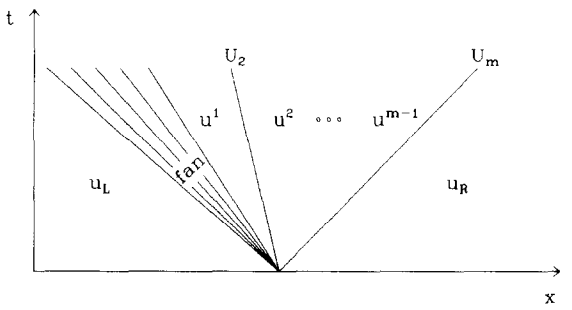

FIG. 5. Form of the solution to the Riemann problem.

The scheme described here suffers from the drawback of being slower than those procedures which have been optimized for particular systems.

Now consider solving the Riemann problem numerically. To each characteristic family $i = 1,..., m$ there will be a possible shock or fan appearing in the solution. The shock or fan may degenerate to zero strength. To begin with one does not know which characteristics lead to shocks and which to fans. This complicates matters in two ways. First the equations that must be solved depend on whether there is a shock or fan; through a fan the characteristic equations are solved while across a shock the Rankine–Hugoniot jump conditions hold. Second, the number of equations varies with the number of shocks, as will be seen shortly. Suppose for the moment that one knows which characteristics form shocks and which form fans. Let n_s be the number of shocks in the solution. The unknowns to be solved for are:

(1) $\{\mathbf{u}^k : k = 1,..., m-1\}$ The constant states which separate k-waves (fan or shock) from $(k+1)$-waves. Define $\mathbf{u}^0$ to be $\mathbf{u}_L$ and $\mathbf{u}^m$ to be $\mathbf{u}_R$.

(2) $\{U_{i_k} : k = 1,..., n_s\}$ The velocities of the shocks which occur on characteristics i_k, $i_k \in (1, 2,..., m)$.

The equations to solve are of the form

(1) The jump conditions across each shock (on characteristic k say)

$$[\mathbf{f}(\mathbf{u}^{k+1}) - \mathbf{f}(\mathbf{u}^k)] - U_k[\mathbf{u}^{k+1} - \mathbf{u}^k] = 0.$$

(2) The characteristic equations through each expansion *fan* (on characteristic k)

$$\mathbf{a}_i(\mathbf{u})^T \frac{d\mathbf{u}}{dt} = 0 \qquad \text{on } C_i, \qquad i = 1, 2,..., k-1, k+1,..., m,$$

where C_i is the characteristic which passes through the k-fan. A simple approximation to this equation is

$$\mathbf{a}_i(\tfrac{1}{2}(\mathbf{u}^{k+1} + \mathbf{u}^k))^T(\mathbf{u}^{k+1} - \mathbf{u}^k) = 0.$$

The number of equations and the number of unknowns are both equal to $m(m-1)+n_s$. In practice it was found necessary to use a more accurate formula for the characteristic equations which are calculated through strong expansion fans (since the solution changes appreciably through the fan). This is done by adding additional states through the fan. If there is a fan between the states $\mathbf{u}^{k-1}$ and $\mathbf{u}^k$ denote the extra states by $\mathbf{u}^{k_v}$, $v=1, 2,..., k_e$. The number of extra states k_e is determined by the angle of opening of the fan. The states are positioned through the fan along the lines

$$x/t = \alpha_v c_k(\mathbf{u}^{k-1}) + (1-\alpha_v)\, c_k(\mathbf{u}^k),$$

where $\alpha_v = v/(k_e+1)$ for the vth extra state. The extra equations that are solved are

$$\mathbf{a}_i\left(\tfrac{1}{2}(\mathbf{u}^{k_{v+1}} + \mathbf{u}^{k_v})\right)^{\mathrm{T}}(\mathbf{u}^{k_{v+1}} - \mathbf{u}^{k_v}) = 0, \qquad i = 1, 2,..., k-1, k+1,..., m$$

$$c_k(\mathbf{u}^{k_v}) = \alpha_v c_k(\mathbf{u}^{k-1}) + (1-\alpha_v)\, c_k(\mathbf{u}^k).$$

In these expressions we have defined

$$\mathbf{u}^{k_0} = \mathbf{u}^{k-1}, \qquad \mathbf{u}^{k_{k_e+1}} = \mathbf{u}^k.$$

Notice that Δt appears nowhere in the equations. This is to be expected as the solution to the Riemann problem is self-similar in variable x/t.

These equations are solved by Newton's method. Table I gives some results for a particular Riemann problem. This is the problem solved in Example 1 of the fourth section. The solution consists of an expansion fan, a contact discontinuity, and a shock. There are two interior constant states. In the table the values of the density, momentum and energy for these constant states are given. The state between the fan and contact discontinuity is denoted by a subscript 2, and the state between the contact and the shock by a subscript 3. Results are given for 0, 1, 2, 3 or 4 extra states through the fan.

TABLE I

Results from the Riemann Solver

Riemann problem					
	Calculated				
True	$k_e=0$	$k_e=1$	$k_e=2$	$k_e=3$	$k_e=4$
$\rho_2=.4263$	.4746	.4397	.4324	.4298	.4285
$\rho_3=.2656$	.2654	.2658	.2657	.2657	.2656
$m_2=.3954$	.4399	.4084	.4014	.3988	.3976
$m_3=.2463$	.2460	.2469	.2466	.2465	.2464
$E_2=.9412$	.9610	.9485	.9447	.9432	.9425
$E_3=.8720$	.8712	.8735	.8729	.8726	.8724

4. Computational Results

4.1. *The Equations of Gas Dynamics*

In this section results are presented for the numerical solution of the equations of gas dynamics. The three components of $\mathbf{u}$ have the more common names of density, momentum and energy.

$$\mathbf{u} = \begin{bmatrix} \rho \\ m \\ E \end{bmatrix} \qquad \begin{aligned} \rho &= \text{density} \\ m &= \text{momentum} \\ E &= \text{Energy.} \end{aligned}$$

The flux function $\mathbf{f}$ for an inviscid polytropic gas is

$$\mathbf{f}(\mathbf{u}) = \begin{bmatrix} m \\ m^2/\rho + p \\ m/\rho(E+p) \end{bmatrix},$$

where the pressure p and velocity u are defined as

$$p = (\gamma - 1)[E - \tfrac{1}{2}m^2/\rho], \qquad \gamma = 1.4$$
$$u = m/\rho.$$

The computer code requires expressions for the coefficients which appear in the characteristic equations. These can be obtained as follows. First the Jacobian matrix is determined.

$$\mathbf{f}_{\mathbf{u}}(\mathbf{u}) = \begin{bmatrix} 0 & 1 & 0 \\ \tfrac{1}{2}(\gamma - 3)\, u^2 & (3-\gamma)\, u & \gamma - 1 \\ -\gamma m E/\rho + (\gamma - 1)\, m^3/\rho^3 & \gamma E/\rho - \tfrac{3}{2}(\gamma - 1)\, m^2/\rho^2 & \gamma u \end{bmatrix}.$$

The eigenvalues and left eigenvectors of this matrix can be calculated in a straightforward manner:

$$c_1 = u - a, \qquad \mathbf{a}_1 \equiv \begin{bmatrix} a_{11} \\ a_{12} \\ a_{13} \end{bmatrix} = \begin{bmatrix} -u(\frac{1}{2}u + a/(\gamma - 1)) \\ u + a/(\gamma - 1) \\ -1 \end{bmatrix}$$

$$c_2 = u, \qquad \mathbf{a}_2 \equiv \begin{bmatrix} a_{21} \\ a_{22} \\ a_{23} \end{bmatrix} = \begin{bmatrix} -\gamma E/\rho - \frac{1}{2}(\gamma + 1)\, u^2 \\ u \\ -1 \end{bmatrix}$$

$$c_3 = u + a, \qquad \mathbf{a}_3 \equiv \begin{bmatrix} a_{31} \\ a_{32} \\ a_{33} \end{bmatrix} = \begin{bmatrix} -u(\frac{1}{2}u - a/(\gamma - 1)) \\ u - a/(\gamma - 1) \\ -1 \end{bmatrix}.$$

TABLE II

Shock Tube Initial Conditions

$t=0$	$x \leqslant .5$	$x \geqslant .5$
ρ	1.0	.125
m	0.0	0.0
E	2.5	.25

The speed of sound a is defined by

$$a^2 \equiv \gamma(\gamma - 1)[E/\rho - \tfrac{1}{2}u^2].$$

Four examples have been chosen to illustrate the performance of the computer code. The first example is the solution of a Riemann problem. The second example

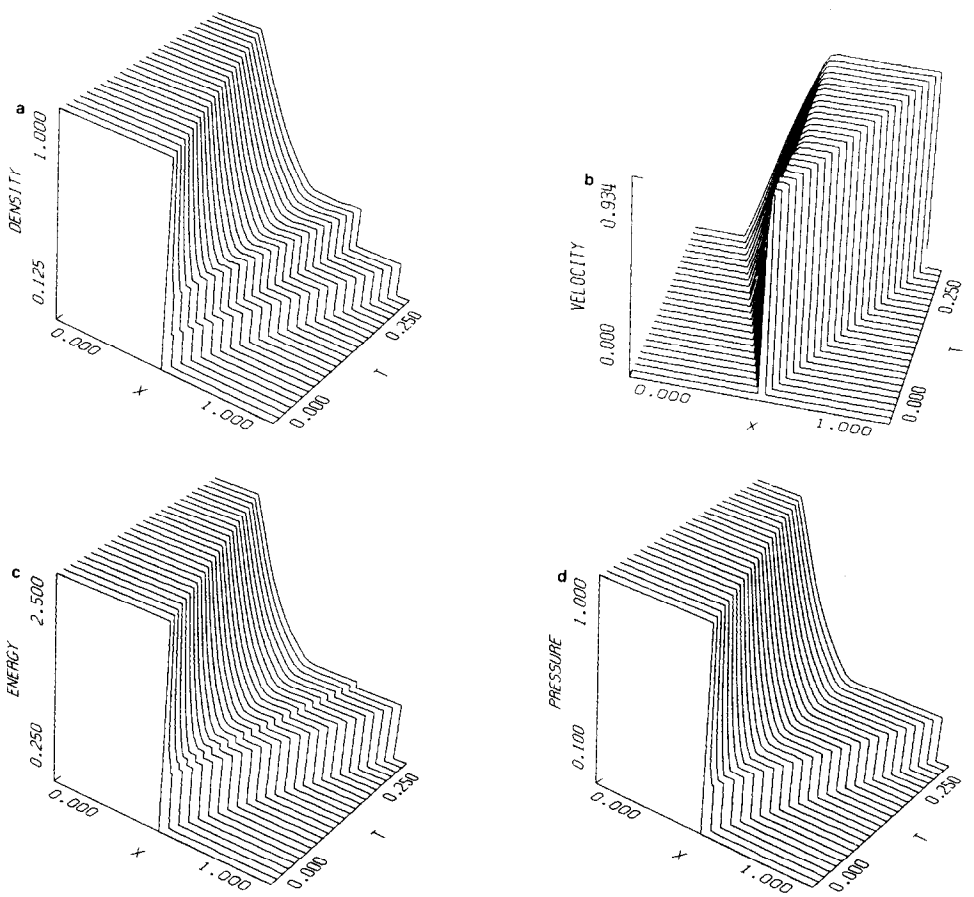

FIG. 6. Shock tube—Time evolution, $\Delta t = .01$, $n = 40$, $n_c = 3$ to 24.

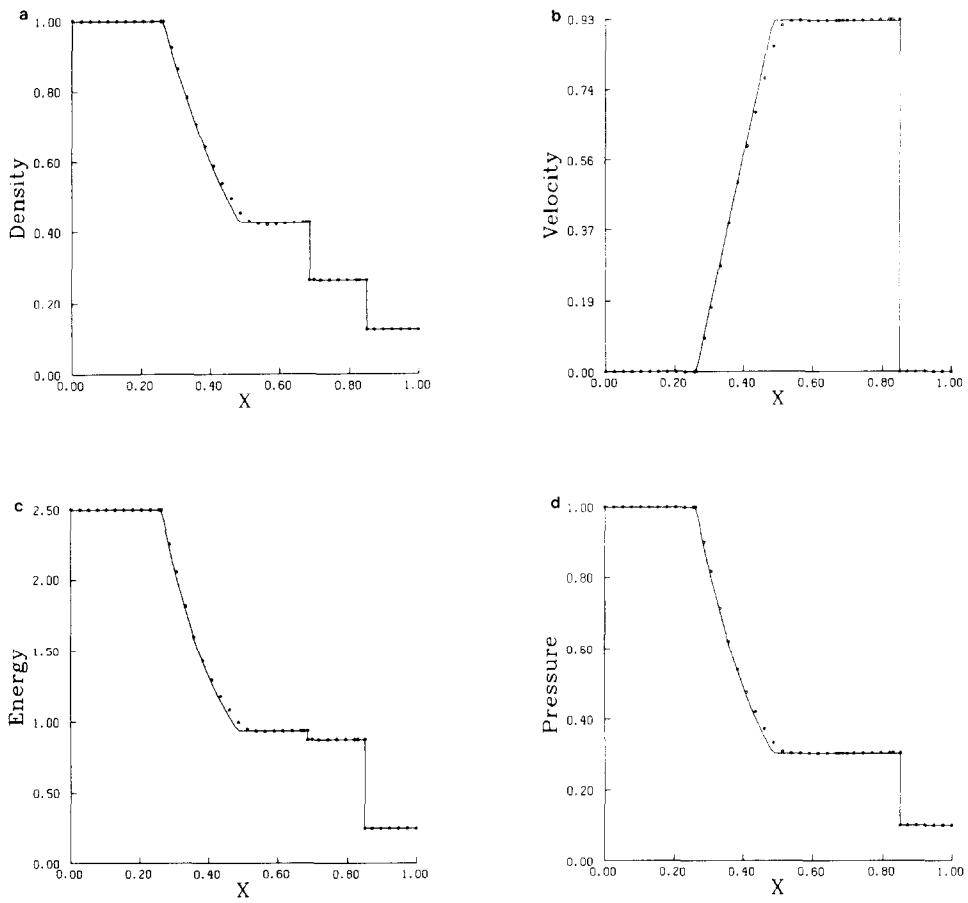

FIG. 7. Shock tube—Comparison to exact solution $t = .2$, $\Delta t = .01$, $n = 40$, $n_c = 10$. Calculated ($\cdots$); exact (——).

shows the collision of two shocks of equal strength. In these first two numerical tests the results are compared to the exact solution. As a third example the formation of a shock is shown. A more complicated problem of interacting shocks, fans and contact discontinuities is given as the final example. Examples 3 and 4 are compared to the results obtained using a more standard finite difference code with many points.

4.2. *Example 1: Shock Tube*

This first example is taken from the paper by Sod [12]. This Riemann problem has become a standard test case. The initial conditions are given in Table II.

The solution for times greater than zero consists of a shock wave travelling to the right followed by a contact discontinuity and a rarefaction wave. The density and

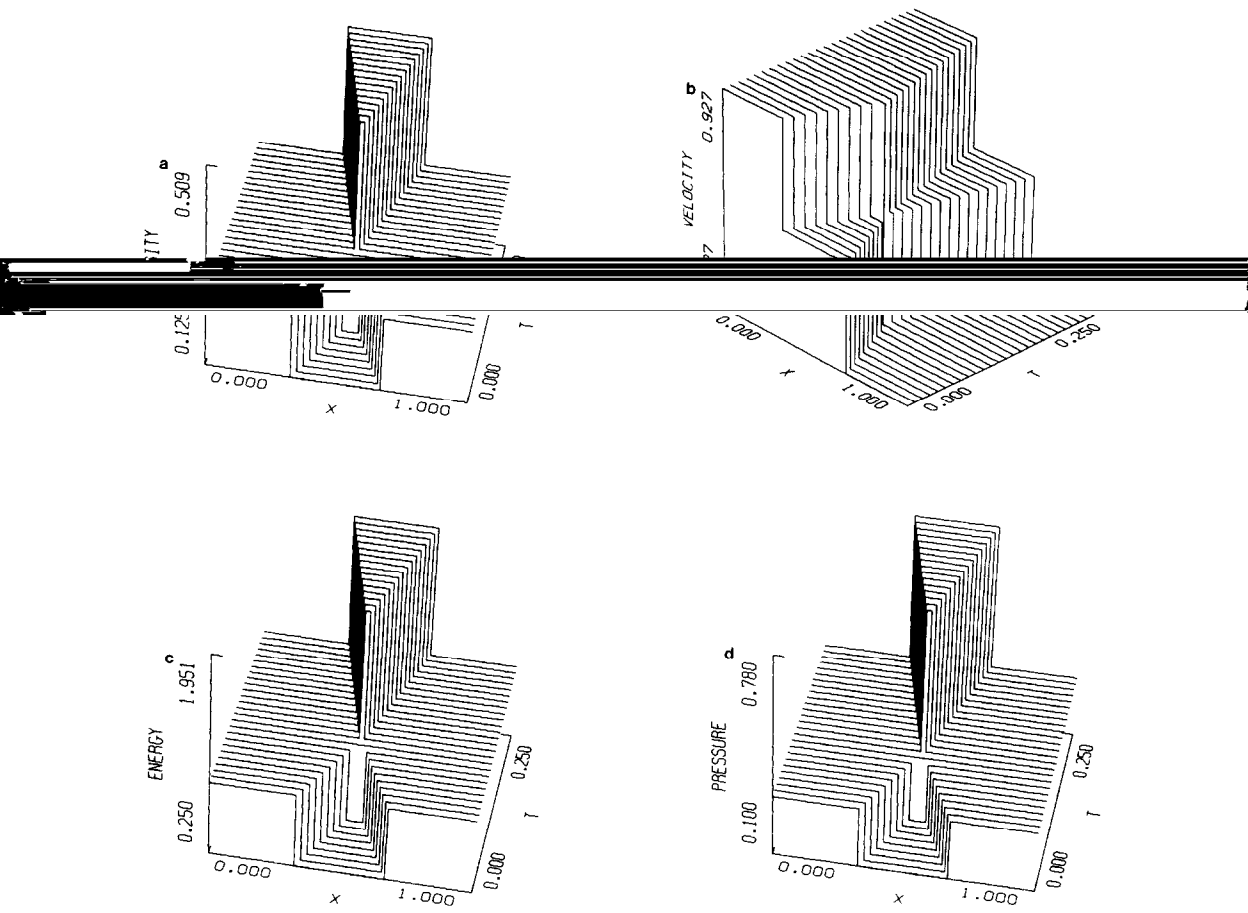

FIG. 8. Shock collision—Time evolution $\Delta t = .01$, $n = 30$, $n_c = 5$ to 11.

energy (and momentum) are discontinuous across the contact, while the velocity and pressure are not. The time evolution of ρ, u, E and p is shown in Fig. 6. Indicated below the figure are the time step Δt, the number of points on the fixed grid n, and the number of characteristic points n_c. n_c is given as a range of values. This range indicates the maximum and minimum number of characteristic points that were needed over the entire run. Initially there is only one *group* of characteristic points. As the discontinuities separate, the group becomes larger and the number of characteristic points increases. When the smooth regions in this group becomes large enough the group splits and the number of characteristic points decreases. By the final time shown there are three groups. Comparison to the exact solution is made at time $t = 0.2$ for $n = 40$ in Fig. 7.

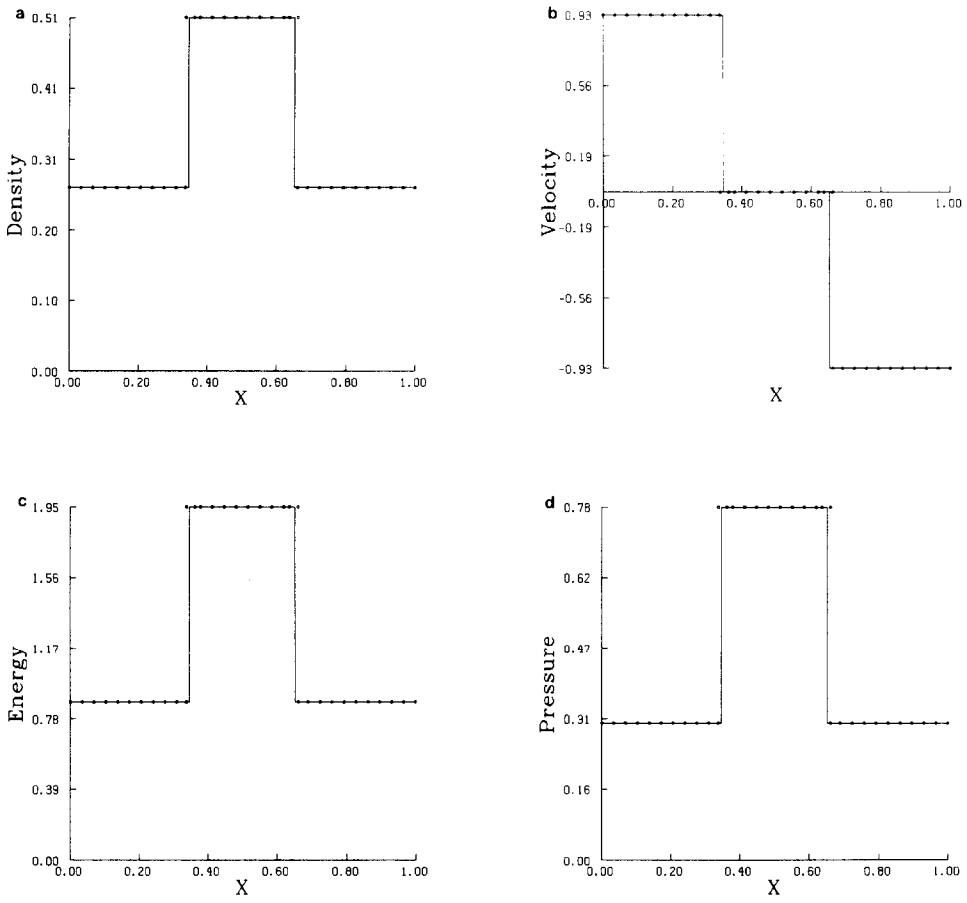

FIG. 9. Shock collision—Comparison to exact solution $t = .25$, $\Delta t = .01$, $n = 30$, $n_c = 7$. Calculated ($\cdots$); exact (——).

4.3. *Example 2: Shock Collision*

This example was chosen to demonstrate how the program handles the collision of shocks. In particular we look at the collision of two shocks of equal strength travelling in opposite directions. The initial conditions are given in Table III.

Figure 8 shows the numerical solution proceeding in time. A comparison is made with the true solution at time $t = 2.5$ in Fig. 9. At this time the shocks have already collided and are now moving apart.

4.4. *Example 3: Shock Formation*

This numerical experiment was performed to see how a smooth profile can steepen up to form a shock (Fig. 10). For the first few steps there are no charac-

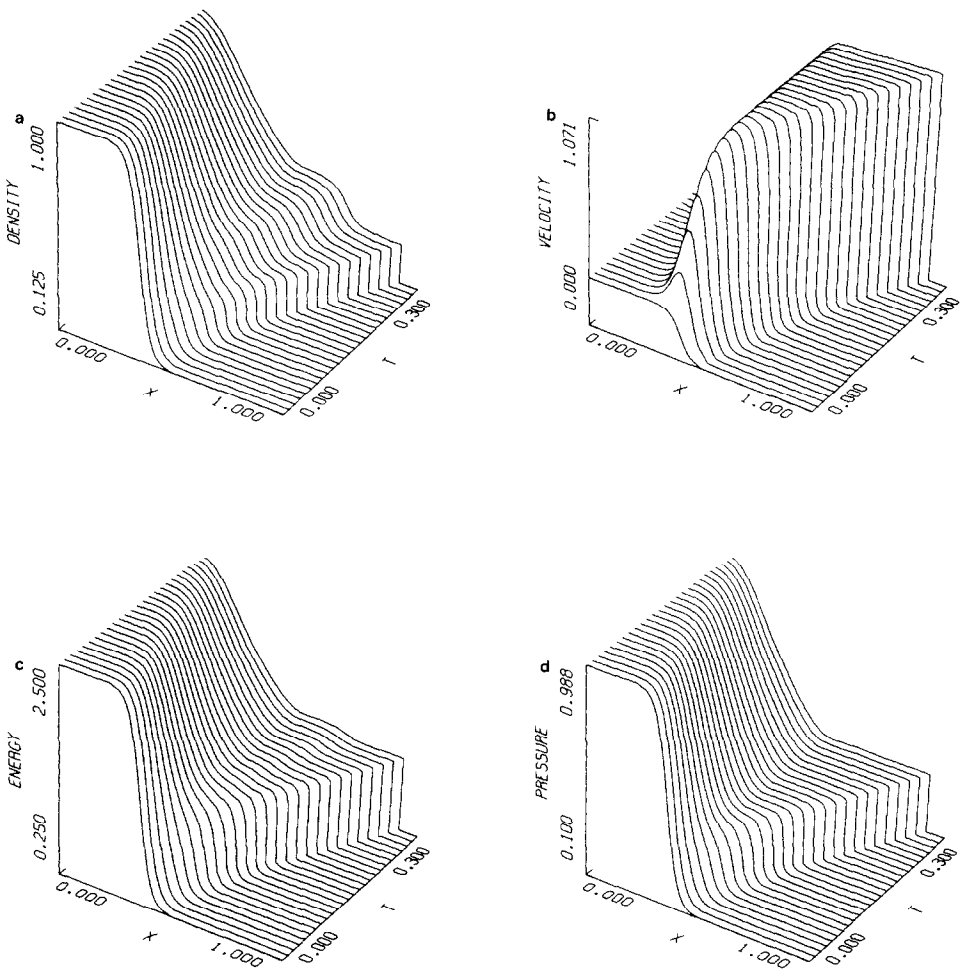

FIG. 10. Shock formation—Time evolution $\Delta t = .004$, $n = 81$, $n_c = 1$ to 9.

TABLE III

Shock Collision Initial Conditions

$t = 0$	$x \leqslant .33$	$.33 \leqslant x \leqslant .67$	$x \geqslant .67$
ρ	.2656	.125	.2656
m	.2463	0.0	−.2463
E	.8720	.25	.8720

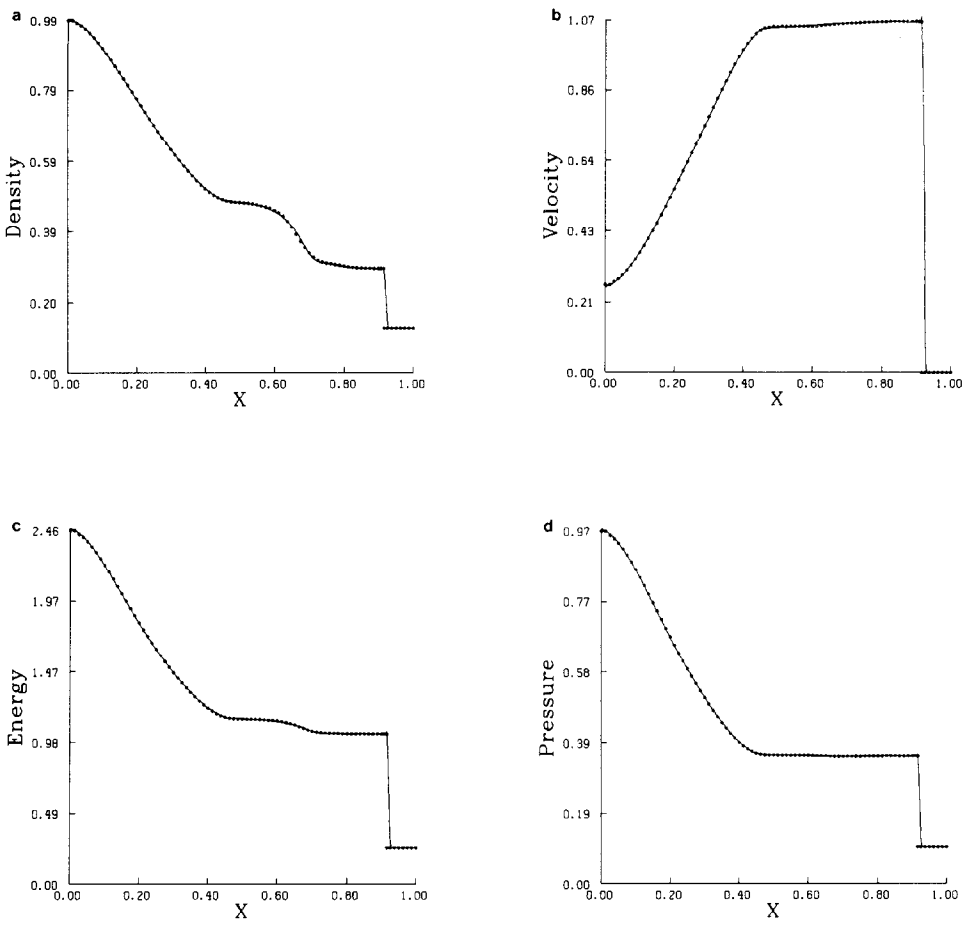

FIG. 11. Shock formation—Comparison to LW solution. Calculated (···) $t = .3$, $\Delta t = .004$, $n = 81$, $n_c = 4$. *LW* (——) $t = .3$, $\Delta t = .01$, $n = 401$, $\nu = 1.5$.

teristic points present. Once the shock starts to form, characteristic points are put in. Eventually the solution develops a jump. Many shock tracking codes would probably not be able to handle a shock formation problem since they require a priori knowledge of the positions of the shocks. The code developed here is well suited for this problem.

This solution is compared to the result obtained using the Lax–Wendroff method on a fixed grid with 401 points (Fig. 11). Artificial viscosity of the type developed by Lapidus [7] is used in the Law–Wendroff solution. The value of this artificial viscosity is given as ν. It is seen that the method accurately predicts the shock development.

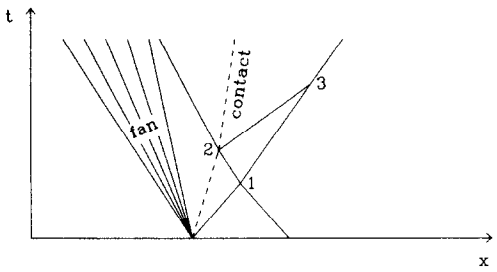

FIG. 12. Shock interactions.

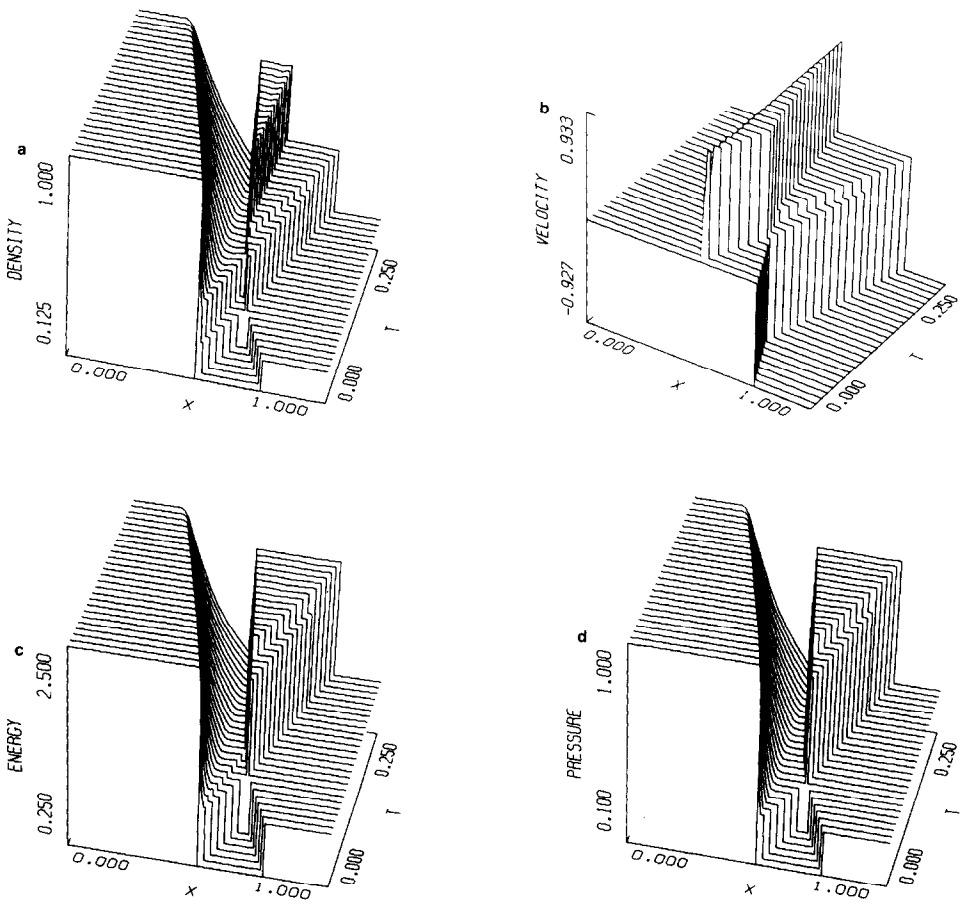

FIG. 13. Shock interactions—Time evolution $\Delta t = .005$, $n = 80$, $n_c = 5$ to 28.

TABLE IV

Initial Conditions for Shock Interactions

ρ	1.0	.125	.2656
m	0.0	0.0	−.2463
E	2.5	.25	.8720

4.5. *Example 4: Interactions*

In this final example a shock moving from the right hits the flow that is generated by the shock tube problem of Example 1. The initial conditions are given in Table IV.

The resulting interactions of shocks and contact discontinuities are fairly complicated and are shown schematically in the x–t diagram of Fig. 12. This should be helpful to follow the numerical solution of Fig. 13.

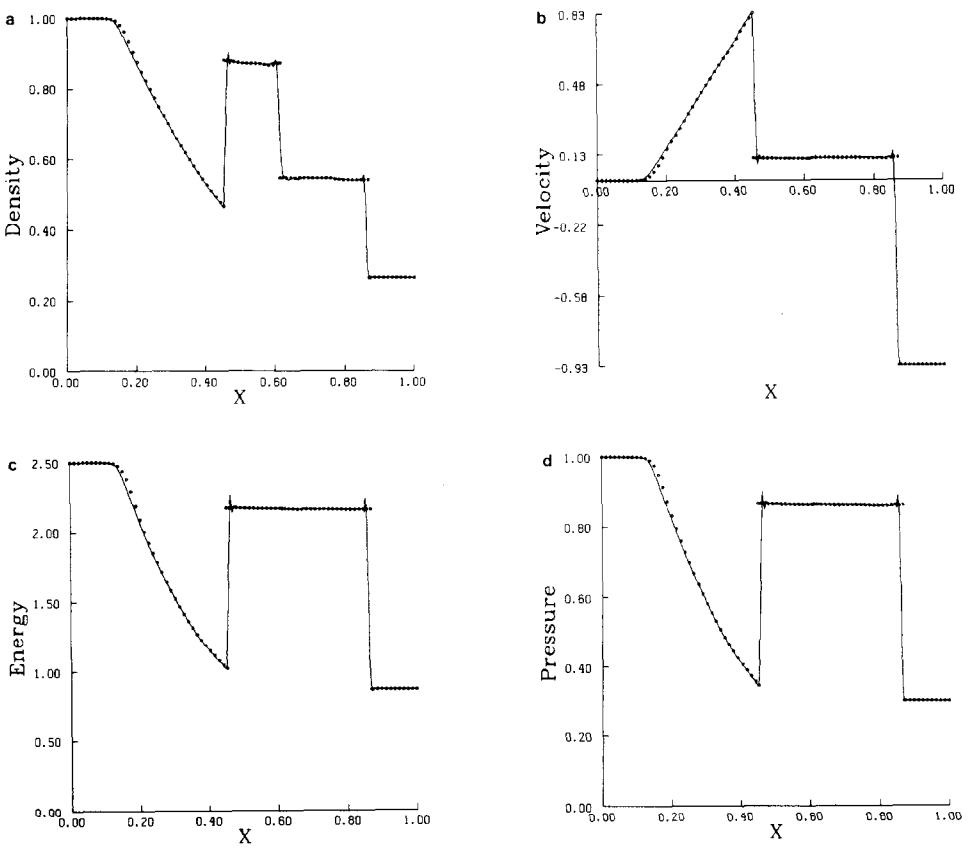

FIG. 14. Shock interactions—Comparison to LW Solution. Calculated (···) $t = .3$, $\Delta t = .005$, $n = 80$, $n_c = 11$. LW (——) $t = .3$, $\Delta t = .001$, $n = 401$, $v = 1.5$.

At the point marked (1) on the diagram the isolated shock which is moving in from the right side of the diagram hits the shock which began at $x = 0.5$ and is moving to the right. These shocks are of equal strength. They pass through each other after being refracted. The one shock continues to move to the left until it hits the contact discontinuity at (2) on the diagram. When the shock and contact hit they pass through each other with some refraction. There is also weak reflected shock which is generated from this collision. This reflected shock moves to the right and catches up with the other shock at (3).

Again the solution is compared to the result using Lax–Wendroff and a large number of points (Fig. 14).

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